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VORTEX MOTIONS OF SOLID MEDIA IN DYNAMIC PROBLEMS OF THE ELASTICITY THEORY

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Within the framework of a problem of constructing a mathematical theory for vortex motions of solid media, we investigated a number of problems of the dynamic elasticity theory with application of numerical and analytical approaches. The solution for a self-similar problem in the finite form is presented, which we were able to divide uniquely into potential and vortex components.

In the present work an attempt is made to analytically describe vortex structures that appear in dynamic problems of the mechanics on deformable solids. The initial stage of this attempt is of an illustrative nature, i.e., on an example of a numerical solution of comparatively simple problems the presence of vortex structures is shown (see [1-3]). Then the specific problem is considered, a feature of which is the fact that naturally it is possible in this problem to uniquely analytically separate a motion into a vortex motion, namely, solenoidal, and potential. A separation method implies that the solution is obtained by expansion of a general solution into a series of certain functions, and here one part of the terms in the series corresponds to a potential component of the solution, while another part, to a vortex component. It is possible to sum the series and to present the vortex and potential components of the general solution in finite form.

1. Problems of Collision. We consider boundary-value axisymmetric problems as a normal collision of a cylinder of radius r_0 and height h' with an absolutely stiff plane (problem 1) and as a coaxial collision, with a round plate of radius R_0' and thickness H' (problem 2) in an elastic arrangement. Assuming a smallness of the rate of collision V_0' (and of deformations and stresses, respectively), the equations of motion in cylindrical coordinates (r, φ, z) can be written in the form of

$$\partial_r\sigma_{rr} + \partial_z\sigma_{rz} (\sigma_{rr} - \sigma_{\varphi\varphi})/r = \rho_i\, \partial^2_{tt} u_r\,, \quad \partial_z\sigma_{zz} + \partial_r\sigma_{rz} + 2\sigma_{rz}/r = \rho_i\, \partial^2_{tt} u_z\,. \tag{1}$$

We use the traditional notation: u_r and u_z are the radial and axial motions; σ_{rr}, σ_{zz}, $\sigma_{\varphi\varphi}$, and σ_{rz} are the components of the stress tensor; ρ_i is the density of the material ($i = 1$ for the cylinder, $i = 2$ for the plate). The relation between the stresses and motions is taken in the following form:

$$\sigma_{rr} = \lambda_i \operatorname{div} \vec{u} + 2\mu_i\, \partial_r u_r\,, \quad \sigma_{\varphi\varphi} = \lambda_i \operatorname{div} \vec{u} + 2\mu_i\, u_r/r\,,$$

$$\sigma_{zz} = \lambda_i \operatorname{div} \vec{u} + 2\mu_i\, \partial_z u_z\,, \quad \sigma_{rz} = \mu_i\, (\partial_z u_r + \partial_r u_z)\,, \tag{2}$$

$$\operatorname{div} \vec{u} = \partial_r u_r + \partial_z u_z + u_r/r\,.$$

The boundary conditions on the contact surface ($z = 0$, $r < r_0$) for problem 1 [1, 3] look like

$$(u_z)_1 = (\sigma_{rz})_1 = 0\,, \tag{3}$$

and for problem 2 [2], like

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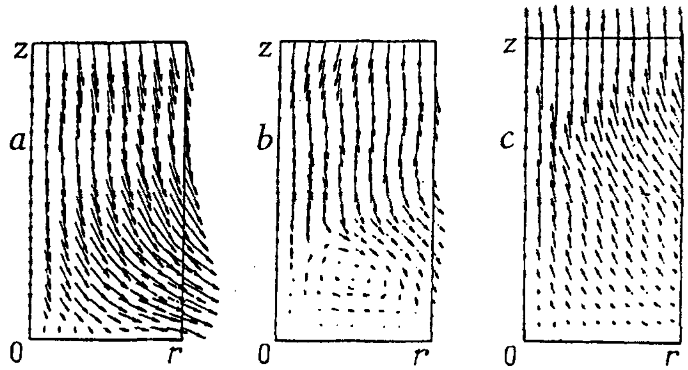

Fig. 1. Patterns of rearrangement of vertical velocity field at collision of an elastic cylinder against a stiff plane (the radius of the cylinder is $r_0 = 5\cdot 10^{-2}$ m; the height is $h' = 10^{-1}$ m).

$$(u_r)_1 = (u_r)_2\,,\ \ (u_z)_1 = (u_z)_2\,,\ \ (\sigma_{zz})_1 = (\sigma_{zz})_2\,,\ \ (\sigma_{rz})_1 = (\sigma_{rz})_2\,. \tag{4}$$

The remaining outer surface is considered to be free of stresses.

The initial conditions (when $t = 0$) are as follows:

$$(u_r)_1 = 0\,,\ \ (\partial_t u_r)_1 = 0\,,\ \ (u_z)_1 = 0\,,\ \ (\partial_t u_z)_1 = -V_0'\,; \tag{5}$$

$$(u_r)_2 = 0\,,\ \ (\partial_t u_r)_2 = 0\,,\ \ (u_z)_2 = 0\,,\ \ (\partial_t u_z)_2 = 0\,. \tag{6}$$

In passing to dimensionless variables, initial equations (1)-(2) and boundary (3) and initial (5) conditions for problem 1 contain the following determining parameters:

$$c_{01} = [\mu_1/(\lambda_1 + 2\mu_1)]^{1/2}\,,\ \ V_0 = V_0'\,[\rho_1/(\lambda_1 + 2\mu_1)]^{1/2}\,,\ \ h = h'/r_0\,. \tag{7}$$

In problem 2 to these parameters we add the following ones:

$$c_{02} = [\mu_2/(\lambda_2 + 2\mu_2)]^{1/2}\,,\ \ \xi = \rho_2/\rho_1\,,\ \ H = H'/r_0\,,\ \ R_0 = R_0'/r_0\,. \tag{8}$$

The above-formulated boundary-value problems are solved numerically by a straight-through method with the help of an explicit finite-difference "leap-frog"-type circuit (by Lax–Vendroff) using a consistent procedure of smoothing, which introduces into the circuit the majorant properties, being not included in it initially.

Dynamic problem 1 for an aluminum cylinder is solved at the following values of the determining parameters (7): $c_{01} = 0.53$; $V_0 = 10^{-2}$; $h = 2$ for the time interval $t < t_1$, where t_1 is the recoil time at which all the points of the contact boundary of a striker are separated from a barrier; this is the instant at which the conditions [4] are satisfied:

$$F(t) = 2\pi \int_0^{r_0} \sigma_{zz}(r, 0, t)\, r dr = 0\,,$$

$$V_m = \left(2\pi \int_0^{h'} \int_0^{r_0} \rho_1\, v r dr dz\right) \Big/ \left(2\pi \int_0^{h'} \int_0^{r_0} \rho_1\, r dr dz\right) > 0\,. \tag{9}$$

Calculations, carried out in the elastic approximation, show that the recoil time in this case is approximately equal to the time of arrival of a wave reflected from a free cylinder end, propagating at the plug velocity, $t_1 = 2h'/c_1$ (thus, for the aluminum striker: $c_1 = 5.5\cdot 10^3$ m/sec, $h' = 10^{-1}$ m, $t_1 \approx 36$ μsec). For this time of double

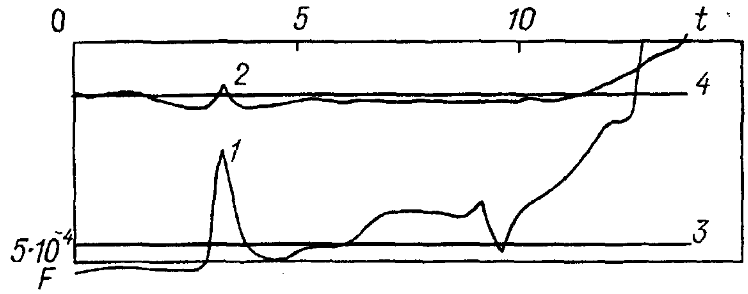

Fig. 2. Change in forces F of interaction on a contact surface in an elastic cylinder-stiff plane system with time t for variants 1 and 2 (lines 1 and 2, respectively). F, N; t μsec.

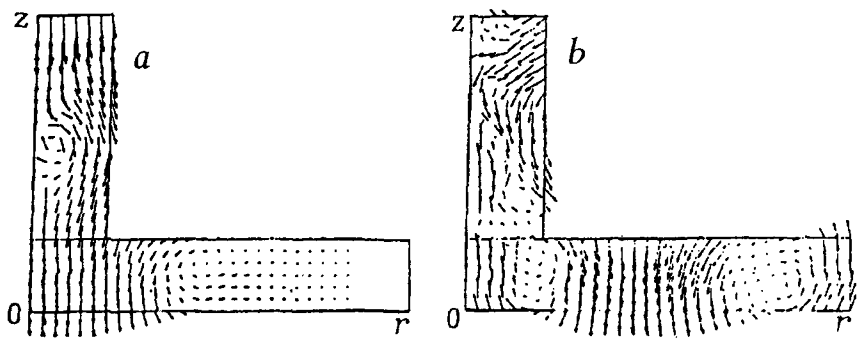

Fig. 3. Patterns of the vector velocity field at collision of elastic cylinder and plate for two time instants $t = 5.5$ and 11 μsec (a and b, respectively: the radius of the cylinder is $r_0 = 10^{-2}$ m and that of the plate is $R_0 = 5 \cdot 10^{-2}$ m; the height of the cylinder is $h' = 3 \cdot 10^{-1}$ m; of the plate it is $H' = 5 \cdot 10^{-2}$ m).

passage of the wave over the cylinder length a velocity field rearranges from vertical directed downward to the barrier, to virtually completely vertical directed upward. The process of rearrangement, observed over the vector velocity field, occurs in the following manner. At the initial stage of interaction with the stiff plane the velocity vectors in the lower half of the cylinder, adjacent to the barrier, move apart as a fan from the symmetry axis to the lateral surface (see Fig. 1a). Then a gradual change begins in the flow: near the contact end it acquires the closed toroidal vortex type with a twist directed to the symmetry axis, while in the upper part the flow remains still nearly vertical directed downward (see Fig. 1b). Further the volumetric vortex structure rises up to the free end, and the vector field by the instant of recoil is completely rearranged (see Fig. 1c).

Dynamic problem 2 for aluminum cylinders (strikers) and a round plate (a target) is solved at the following values of determining parameters (7) and (8): $c_{01} = c_{02} = 0.53$; $V_0 = 7 \cdot 10^{-3}$; $h = 0.3$; $H = 1$; $R_0 = 5$ (variant 1), and $h = 0.6$; $H = 2$; $R_0 = 10$ (variant 2) for the time interval $t < t_1$, where t_1 is the recoil time determined from criterion (9).

From the approximate theory constructed in [2] it follows that for time instants close to the initial $t < h/c_2$ (c_2 is the plug velocity in the plate), a force on the contact surface must be constant and determined by the relation

$$F = \rho \frac{c_2 V_0}{2} \pi r_0^2. \tag{10}$$

Actually, as the calculations show, when $t < 2$ μsec ($h/c_2 = 1.8$ μsec) the obtained forces $F(t)$ on the contact remain constant (see curves 1 and 2 in Fig. 2), and here the greater the difference in the radii of the striker and plate ($r_0 = 10^{-2}$ m, $R_0' = 5 \cdot 10^{-2}$ m, i.e., $R_0 = 5$ (variant 1) and $r_0 = 5 \cdot 10^{-3}$ m, $R_0' = 5 \cdot 10^{-2}$ m, i.e., $R_0 = 10$ (variant 2)), the better the coincidence of the calculated force $F(t)$ with the value given by formula (10) (compare curves 1 and 2 with lines 3 and 4 in Fig. 2, respectively). Moreover, for variant 2 the force on the contact remains close to the theoretical value almost to the recoil instant.

In Fig. 3 for two time instants $t = 5.5$ and 11 μsec we present instantaneous patterns of the velocity fields in the right half of the cross section plane $(r\,0\,z)$ for variant 1. Intense toroidal vortex structures, originated near the angular points over the perimeter of the circular contact region in both the cylinder and the plate, further propagate upward, i.e., to the free cylinder end and laterally, namely, to the lateral surface of the round plate, with rotation directions in them being opposite (see Fig. 3a). In the course of subsequent evolution of the process, i.e., of complex wave and vortex interactions inside of the objects and on boundaries, secondary vortex structures originate and gradually the velocity field in the striker and in the part of the plate located directly under the striker rearranges vertically and directed upward (see Fig. 3b). By the time instant $t \approx 13$ μsec the flow parameters take the values that satisfy criterion (9), and a recoil occurs.

Completing the first section it should be noted that, on the one hand, vortex motions of solid media (as our computational experiments showed [3]) generate under similar actions not only in elastic objects but also in elasticoplastic and viscoelastic ones; on the other hand, the vortex motions take place also in numerous other dynamic processes in the mechanics of deformable solids, and (as the authors informed us kindly) they were observed at collision of different cylindrical objects [5, 6], on breaking through barriers [7], under the action of a die on a layer [8], etc.

2. Problem of Motion of a Wedge in an Elastic Plane. As an example of the problem in which it is possible in a natural manner to uniquely divide a motion into vortex and potential ones, we consider the deformation of an elastic plane at the prescribed initial velocity in the wedge-shaped region. At the initial time instant $t = 0$, in the infinite wedge-shaped region with an arbitrary aperture angle $(\alpha_2 - \alpha_1)$ between faces-sides (α_1 is the angle between the first side and the abscissa axis x, while α_2, between the second side and the abscissa axis x) the constant velocity $\vec{V}_0' = (V_x', V_y')$ is prescribed. Assuming a smallness of the velocity $\vec{V}_0'$, the equations of motion in a polar coordinate system (r, φ) have the form

$$\partial_r \sigma_{rr} + \partial_\varphi \sigma_{r\varphi}/r + (\sigma_{rr} - \sigma_{\varphi\varphi})/r = \rho \partial^2_{tt} u_r ,$$
$$\partial_\varphi \sigma_{\varphi\varphi}/r + \partial_r \sigma_{r\varphi} + 2\sigma_{r\varphi}/r = \rho \partial^2_{tt} u_\varphi , \tag{11}$$

where connections between the stresses and motions are determined by the relations

$$\sigma_{rr} = \lambda \operatorname{div} \vec{u} + 2\mu \partial_r u_r , \quad \sigma_{\varphi\varphi} = \lambda \operatorname{div} \vec{u} + 2\mu u_r/r ,$$
$$\sigma_{r\varphi} = \mu\,(\partial_\varphi u_r/r + \partial_r u_\varphi - u_r/r) , \quad \operatorname{div} \vec{u} = \partial_r u_r + \partial_\varphi u_\varphi/r + u_r/r . \tag{12}$$

Initial conditions (at $t = 0$) are as follows

$$u_r = u_\varphi = 0 , \quad \partial_t u_r = \partial_t u_\varphi = 0 \quad \text{when} \quad \varphi \in \,]0, \alpha_1[\,\cup\,]\alpha_2, 2\pi[\,;$$
$$\partial_t u_r = V_x' \cos\varphi + V_y' \sin\varphi , \quad \partial_t u_\varphi = V_y' \cos\varphi - V_x' \sin\varphi \quad \text{when} \quad \varphi \in \,]\alpha_1, \alpha_2[\,. \tag{13}$$

Since the dimensional scale of length is absent, from problem statement (11)-(13) we can seek a solution in the form of

$$u_r = t f(z, \varphi) , \quad u_\varphi = t g(z, \varphi) , \tag{14}$$

where $z = r/c_1 t$; $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$ is the propagation velocity of volumetric compression.

Initial equations (11)-(12) and initial conditions (13) after nondimensionalization contain the following determining parameters of the problem:

$$c_0 = [\mu/(\lambda + 2\mu)]^{1/2} , \quad V_x = V_x'/c_1 , \quad V_y = V_y'/c_1 , \quad \alpha_1 , \quad \alpha_2 \tag{15}$$

and after substitution of Eq. (14) into (11)-(12) and (13), they take the form

$$(1 - z^2)\, \partial^2_{zz} f + \partial_z f/z - f/z^2 + (c_0/z)^2\, \partial^2_{\varphi\varphi} f +$$

$$+ [(1 - c_0)^2/z]\, \partial^2_{z\varphi}\, g - [(1 + c_0)^2/z^2]\, \partial_\varphi\, g = 0\,,$$

$$[(1 - c_0)^2/z]\, \partial^2_{z\varphi} f + [(1 + c_0)^2/z^2]\, \partial_\varphi f + (c_0^2 - z^2)\, \partial^2_{zz}\, g + \tag{16}$$

$$+ (c_0^2/z)\, \partial_z\, g - c_0^2\, g/z^2 + (\partial^2_{\varphi\varphi}\, g)/z^2 = 0$$

and

$$\lim_{z\to\infty} f(z, \varphi) = \lim_{z\to\infty} g(z, \varphi) = 0 \quad \text{when} \quad \varphi \in\,]0, \alpha_1[\, \cup\,]\alpha_2, 2\pi[\,,$$

$$\lim_{z\to\infty} f(z, \varphi) = V_x \cos\varphi + V_y \sin\varphi\,, \;\; \lim_{z\to\infty} g(z, \varphi) = V_y \cos\varphi - V_x \sin\varphi \tag{17}$$

$$\text{when} \quad \varphi \in\,]\alpha_1, \alpha_2[\,.$$

Solution of formulated self-similar problem (16)-(17) is sought in the form:

$$f(z, \varphi) = u_0(z) + \sum_{n=1}^{\infty} (u_{n,1}(z) \cos \alpha_n \varphi + u_{n,2}(z) \sin \alpha_n \varphi)\,,$$

$$g(z, \varphi) = v_0(z) + \sum_{n=1}^{\infty} (v_{n,1}(z) \sin \alpha_n \varphi + v_{n,2}(z) \cos \alpha_n \varphi)\,. \tag{18}$$

Using the properties of continuity, boundedness, and periodicity of solution (18), after tedious calculations (for this reason they are not presented here) we obtain its finite form, consisting of the potential (p) and solenoidal (s) parts:

$$f(z, \varphi) = f_{\mathrm{p}}(z, \varphi) + f_{\mathrm{s}}(z, \varphi)\,, \;\; g(z, \varphi) = g_{\mathrm{p}}(z, \varphi) + g_{\mathrm{s}}(z, \varphi)\,. \tag{19}$$

The potential components of the solution look like:

$$f_{\mathrm{p}}(z, \varphi) = \sum_{i=1}^{2} \Phi_{\mathrm{p}}(i, \alpha_i)\, G(\tau, \beta_i)\,,$$

$$g_{\mathrm{p}}(z, \varphi) = (V_y \cos\varphi - V_x \sin\varphi)\, \Pi(\varphi, \alpha_1, \varphi_2) - \sum_{i=1}^{2} \Phi_{\mathrm{p}}(i, \alpha_i)\, F(\tau, \beta_i)\,, \tag{20}$$

while the solenoidal (vortex) components, like:

$$f_{\mathrm{s}}(z, \varphi) = \sum_{i=1}^{2} \Phi_{\mathrm{s}}(i, \alpha_i)\, F(\tau, \beta_i) + (V_x \cos\varphi + V_y \sin\varphi)\, \Pi(\varphi, \alpha_1, \alpha_2)\,,$$

$$g_{\mathrm{s}}(z, \varphi) = \sum_{i=1}^{2} \Phi_{\mathrm{s}}(i, \alpha_i)\, G(\tau, \beta_i)\,. \tag{20'}$$

Here

$$\beta_i = \varphi - \alpha_i\,, \;\; \Phi_{\mathrm{s}}(i, \alpha_i) = (-1)^i\, (V_x \cos\alpha_i + V_y \sin\alpha_i)/2\pi\,,$$

$$\Phi_p(i, \alpha_i) = (-1)^i (V_x \sin \alpha_i + V_y \cos \alpha_i)/2\pi ,$$

$$\Pi(\varphi, \alpha_1, \alpha_2) = 1 \quad \text{when} \quad \varphi - \pi \in \,]\alpha_1, \alpha_2, [\, ,$$

$$\Pi(\varphi, \alpha_1, \alpha_2) = 0 \quad \text{when} \quad \varphi - \pi \in \,]0, \alpha_1 [\, \cup \,]\alpha_2, 2\pi [\, , \tag{21}$$

$$F(\tau, \beta_i) = A(\tau) \sin \beta_i + B(\tau, \beta_i) \cos \beta_i - C(\tau, \beta_i) \sin 2\beta_i/2 ,$$

$$G(\tau, \beta_i) = A(\tau) \cos \beta_i - B(\tau, \beta_i) \sin \beta_i + C(\tau, \beta_i) \sin^2 \beta_i ,$$

where τ, $A(\tau)$, $B(\tau, \beta_i)$, and $C(\tau, \beta_i)$ are different with respect to z for the two regions:
when $z \in \,]0, c_0[$

$$\tau = \ln \left\{ [c_0 + (c_0^2 - z^2)^{1/2}]/z \right\}, \quad C(\tau, \beta_i) = [\pi/2 + \arctan (\cos \beta_i / \sinh \tau)]/\cosh \tau ,$$
$$B(\tau, \beta_i) = \arctan [\sin 2\beta_i/(\exp 2\tau + \cos 2\beta_i)] - \beta_i , \quad A(\tau) = \tau - \tanh \tau ; \tag{22}$$

when $z \in \,]c_0, \infty[$

$$\tau = \arccos (c_0/z) ,$$

$$A(\tau) = 0 , \quad B(\tau, \beta_i) = \pi , \quad C(\tau, \beta_i) = 0 \quad \text{when} \quad \beta_i \in \,]-\pi, \tau - \pi/2[\, ,$$

$$A(\tau) = 0 , \quad B(\tau, \beta_i) = 0 , \quad C(\tau, \beta_i) = \pi/\cos \tau \quad \text{when} \quad \beta_i \in \,]\tau - \pi/2, \pi/2 - \tau[\, , \tag{23}$$

$$A(\tau) = 0 , \quad B(\tau, \beta_i) = -\pi , \quad C(\tau, \beta_i) = 0 \quad \text{when} \quad \beta_i \in \,]\pi/2 - \tau, \pi[\, .$$

The corresponding values for the solenoidal velocity components $(v_r)_s$ and $(v_\varphi)_s$ take the form:

$$(v_r)_s = \sum_{i=1}^{2} \Phi_s(i, \alpha_i) \, [F(\tau, \beta_i) + P(\tau) \, \partial_\tau F(\tau, \beta_i)] +$$
$$+ (V_x \cos \varphi + V_y \sin \varphi) \, \Pi(\varphi, \alpha_1, \alpha_2) , \tag{24}$$
$$(v_\varphi)_s = \sum_{i=1}^{2} \Phi_s(i, \alpha_i) \, [G(\tau, \beta_i) + P(\tau) \, \partial_\tau G(\tau, \beta_i)] .$$

Here $\Phi_s(i, \alpha_i)$, $F(\tau, \beta_i)$, and $G(\tau, \beta_i)$ are determined by relations (21)-(23) and by

$$\partial_\tau F(\tau, \beta_i) = D(\tau) \sin \beta_i + E(\tau, \beta_i) \cos \beta_i - Q(\tau, \beta_i) \sin 2\beta_i/2 ,$$
$$\partial_\tau G(\tau, \beta_i) = D(\tau) \cos \beta_i - E(\tau, \beta_i) \sin \beta_i + Q(\tau, \beta_i) \sin^2 \beta_i , \tag{25}$$

where τ, $P(\tau)$, $D(\tau)$, $E(\tau, \beta_i)$, and $Q(\tau, \beta_i)$ are different with respect to z for the two regions:
when $z \in \,]0, c_0[$

$$\tau = \ln \left\{ [c_0 + (c_0^2 - z^2)^{1/2}/z \right\}, \quad P(\tau) = \operatorname{ctanh} \tau ,$$

$$D(\tau) = 1 - \cosh^{-2} \tau , \quad Q(\tau, \beta_i) = -\tanh \tau C(\tau, \beta_i) ,$$

$$E(\tau, \beta_i) = \sin 2\beta_i \left\{ (\sinh^2 \tau + \cos^2 \beta_i)^{-1} - \right.$$

$$- 4 \exp 2\tau \, [(\exp 2\tau + \cos 2\beta_i)^2 + \sin^2 2\beta_i]^{-1}\Big\}/2 \, ; \qquad (26)$$

when $z \in \,]c_0, \infty[$

$$\tau = \arccos (c_0/z) \, , \quad P(\tau) = - \operatorname{ctan} \tau \, , \quad D(\tau) = E(\tau, \beta_i) = 0 \, ,$$

$$Q(\tau, \beta_i) = 0 \quad \text{when} \quad \beta_i \in \,]-\pi, \tau - \pi/2[\, \cup \,]\pi/2 - \tau, \pi[\, , \qquad (27)$$

$$Q(\tau, \beta_i) = \pi \tan \tau \cos^{-1} \tau \quad \text{when} \quad \beta_i \in \,]\tau - \pi/2, \pi/2 - \tau[\, .$$

The functions of vortices for the motion u and velocity v have the form

$$\operatorname{rot} \vec{u} = \sum_{i=1}^{2} \Phi_s(i, \alpha_i) \, U(\tau, \beta_i) \, , \qquad (28)$$

where $U(\tau, \beta_i)$ take different values with respect to z for the two regions:
when $z \in \,]0, c_0[$

$$U(\tau, \beta_i) = \cosh \tau \, C(\tau, \beta_i)/c_0 \, ; \qquad (29)$$

when $z \in \,]c_0, \infty[$

$$U(\tau, \beta_i) = 0 \quad \text{when} \quad \beta_i \in \,]-\pi, \tau - \pi/2[\, \cup \,]\pi/2 - \tau, \pi[\, ; \qquad (30)$$

$$U(\tau, \beta_i) = \pi/c_0 \quad \text{when} \quad \beta_i \in \,]\tau - \pi/2, \pi/2 - \tau[\, ;$$

$$\operatorname{rot} \vec{v} = \sum_{i=1}^{2} \Phi_s(i, \alpha_i) \, V(\tau, \beta_i) \, , \qquad (31)$$

where $V(\tau, \beta_i)$ take different values with respect to z for the two regions:
when $z \in \,]0, c_0[$

$$\tau = \ln \left\{ [c_0 + (c_0^2 - z^2)^{1/2}/z \right\} , \qquad (32)$$

$$V(\tau, \beta_i) = - \operatorname{ctanh} \tau \cos \beta_i \, [c_0 \sinh \tau \, (\sinh^2 \tau + \cos^2 \beta_i)]^{-1} \, ;$$

when $z \in \,]c_0, \infty[$

$$\tau = \arccos (c_0/z) \, ,$$

$$V(\tau, \beta_i) = \pi \, [\delta(\beta_i - \tau + \pi/2) + \delta(\beta_i + \tau - \pi/2)]/c_0 \, , \quad i = 1, 2 \, . \qquad (33)$$

We note that potential and vortex solutions have a specific feature at the wedge vertex, which is annihilated on their summation.

Before proceeding to graphical interpretation and subsequent analysis of analytical solution (20)-(32) for different values of initial conditions (13), we note that in section 1 the vortex motions of the medium are illustrated by the velocity fields (see Figs. 1 and 3); therefore, first, calculations in [1, 2] were carried out not directly from Eqs. (1)-(2) for motions but by their analogs in velocity variables and, second, for possible comparison of the resulting vortex-type flows with similar flows in hydrodynamics that can be naturally visualized using computer graphing by the velocity fields. Further, in direct study of vortex motions in solid media, for illustrations use is

Variant 1

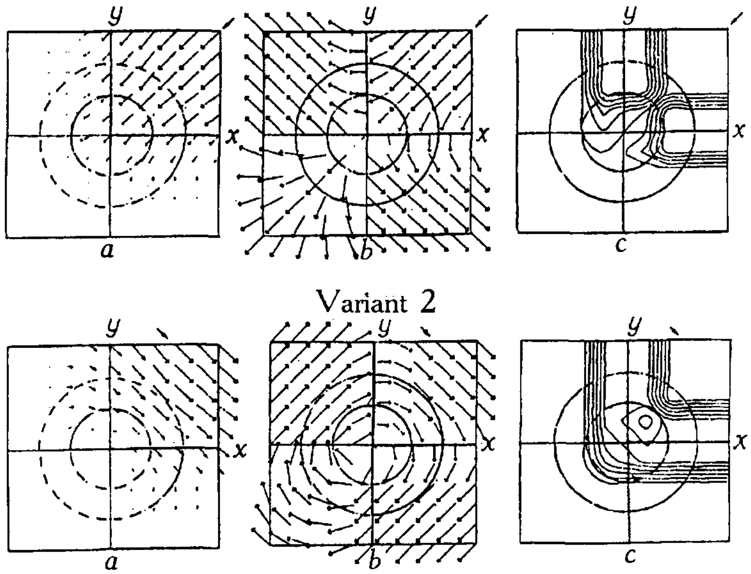

Fig. 4. Patterns of vector fields of total (a), vortex (b), and rot $\vec{u}$ (c) motions with the direction of the initial velocity $\vec{V}$ over the first quadrant bisectrix (variant 1) and over the normal to the first quadrant bisectrix (variant 2).

made of fields of motions, since the stressed-deformable state of solid substances is determined either by a current type of this field or by the entire history of its variation.

We consider solutions of problem (11)-(13) for two sets of determining parameters (2.5): $c_0 = 0.55$; $\alpha_1 = 0$; $\alpha_2 = \pi/2$; $V_y = -10^{-3}$; $V_x = -10^{-3}$ (variant 1) and $V_x = 10^{-3}$ (variant 2).

Variant 1 corresponds to the assignment inside of the first quadrant of the velocity, equal to $|\vec{V}| = \sqrt{2}\cdot 10^{-3}$ and directed along a proper bisectrix; variant 2, to the assignment of the same velocity in the modulus directed to the bisectrix over the normal.

Figure 4 presents the vector fields of motions $\vec{u}$ (a) and of their solenoidal components $\vec{u}_s$ (b), as well as the isolines of the vortex functions rot $\vec{u}$ (c) for variants 1 and 2, respectively, in a square with a center at the coordinate origin (0, 0) and a side equal to 2.9 in the variables $z = r/c_1 t$ (Fig. 4 illustrates the axes of coordinates x and y and the circles of radii $z = c_0$ and 1; arrows indicate the directions of initial motion). From the fields of motions $\vec{u}$ (see Fig. 4a, variants 1 and 2) it is difficult to say anything about the nature of the medium motion apart from its obvious symmetry with respect to the bisectrix $(\alpha_2 - \alpha_1)/2$ for variant 1. From the distributions of the vortex functions rot $\vec{u}$ (see Fig. 4c, variants 1 and 2) it is possible only to establish that the medium motion in both cases is vortical, since there are regions where rot $\vec{u} \neq 0$. Patterns of the fields of solenoidal motions are considerably more informative, since with these it is evident that for the first variant the vortex motion is separated into two vortex flows that are symmetric about the bisectrix (see Fig. 4b, variant 1), while for the second variant, the vortex motion has a unique twisted closed structure, which rotates in a clockwise direction with a center near the angular point of the wedge (see Fig. 4b, variant 2).

In conclusion we note that purely from theoretical and methodical viewpoints the problem investigated is a new section in the dynamics of solid media, since the theory created for vortex flows of solid media is new by itself as are the investigation methods developed on the basis of analytical and numerical approaches.

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NOTATION

x, y, Cartesian coordinates; r, z, φ, radial, axial, and azimuth coordinates; t, time; $\vec{u}(u_r, u_z, u_\varphi)$, motion; $\vec{v}(v_r, v_z, u_\varphi)$, velocity; σ_{rr}, σ_{zz}, σ_{rz}, $\sigma_{r\varphi}$, components of the stress tensor; ρ_i, material density; $V(V_x, V_y)$ and V_0',

V_0, initial velocity (with a prime, dimensional; without a prime, dimensionless); λ_i and μ_i, Lamé parameters; c_i, plug velocity; $c_0 = [\mu/(\lambda + 2\mu)]^{1/2}$; r_0, R_0 and h_0, H_0, radii and heights of the objects; V_m, vertical velocity component of the cylinder mass center; $F(t)$, force on the contact surface of the objects; α_1, α_2, angles between the wedge sides and abscissa axis; $\beta_i = \varphi - \alpha_i$, $f(z, \varphi)$, $g(z, \varphi)$, self-similar solution; f_p, g_p, and f_s, g_s, potential and vortex components of the solution, respectively; $\Phi_p(i, \alpha_i)$, $\Phi_s(i, \alpha_i)$, $\Pi(\varphi, \alpha_1, \alpha_2)$, $F(\tau, \beta_i)$, $G(\tau, \beta_i)$, $A(\tau)$, $B(\tau,\beta_i)$, $C(\tau, \beta_i)$, $E(\tau, \beta_i)$, $Q(\tau, \beta_i)$, $P(\tau)$, $D(\tau)$, $U(\tau, \beta_i)$, $V(\tau, \beta_i)$, auxiliary functions; δ, delta-function. Subscripts: m, mass center; p, potential components; s, vortex components; $i = 1, 2$.

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